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**What do normative indices of  
multidimensional inequality really measure?**

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**Abstract**

We argue that normative indices of multidimensional inequality do not only measure a distribution's extent of inequity (i.e., the gaps between the better-off and the worse-off), but also its extent of inefficiency (i.e., the non-realized mutually beneficial exchanges of goods). We provide a decomposition that allows quantifying these two parts of inequality. Exact formulas of the inequity and inefficiency components are provided for a generic class of social welfare functions. The inequity component turns out to be a two-stage measure, which applies a unidimensional inequality measure to the vector of well-being levels. We critically discuss two prominent transfer principles, viz., uniform majorization and correlation increasing majorization, in the light of the decomposition. A decomposition of inequality in human development illustrates the analysis.

**Keywords:** multidimensional inequality measurement, efficiency, uniform majorization, correlation increasing majorization.

**JEL Classification:** D31, D63, I31

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## 1 Introduction

The normative approach to inequality measurement focuses on the social welfare gain that could be obtained by optimally redistributing the available goods. The greater is this potential welfare gain, the greater is inequality.<sup>1</sup> If there is only one good, as in the case of income inequality, then the only way the ethical observer can increase welfare is by redistributing from better off to worse off individuals, i.e., by improving equity. Hence, in the unidimensional setting, inequality as defined in the normative approach coincides with inequity.

The identity of inequality and inequity vanishes once we move to the multidimensional setting. In addition to equity improvements, now also efficiency improvements become possible: by exchanging goods between individuals with unequal marginal rates of substitution, the ethical observer can increase the well-being levels of all individuals involved. Because normative indices of multidimensional inequality measure the total potential welfare gain, they capture both inequity and inefficiency. It immediately follows that the use of the terms inequality and inequity as synonyms is misleading in the multidimensional setting.

A simple example, with two individuals and two goods, will illustrate the importance of distinguishing between inequity and inefficiency in multidimensional inequality judgments. For the example, we will assume that social welfare is measured using two symmetric Cobb-Douglas functions, one to aggregate individual bundles of goods into well-being levels and one to aggregate individual well-being levels into social welfare.<sup>2</sup> The assumption of a common individual well-being function is of course restricting. Nevertheless, we will adopt it throughout the paper because it is a standard, though often implicit assumption in the literature under examination.<sup>3</sup>

Consider the left-hand panel of Figure 1. Depicted is distribution  $X = (x_1, x_2)$ , where  $x_1 = (70, 10)$  is the bundle of individual 1 and  $x_2 = (10, 70)$  is the bundle of individual 2. Also depicted is distribution  $Y = (y_1, y_2)$ , where  $y_1 = y_2 = (40, 40)$ , the welfare maximizing distribution given the

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<sup>1</sup>The normative approach was pioneered in the unidimensional setting by Dalton (1920), Kolm (1969), Atkinson (1970) and Sen (1973). In the multidimensional setting, the approach was first suggested by Kolm (1977) and further developed by Tsui (1995). See Weymark (2006) for a survey.

<sup>2</sup>This specification is obtained by setting  $\alpha = 1$ ,  $\beta = 1$  and  $r_1 = r_2$  in equations (5) and (6) below.

<sup>3</sup>An exception is the study by Decancq, Fleurbaey and Maniquet (2012), which incorporates heterogeneous preferences into a multidimensional poverty measure.

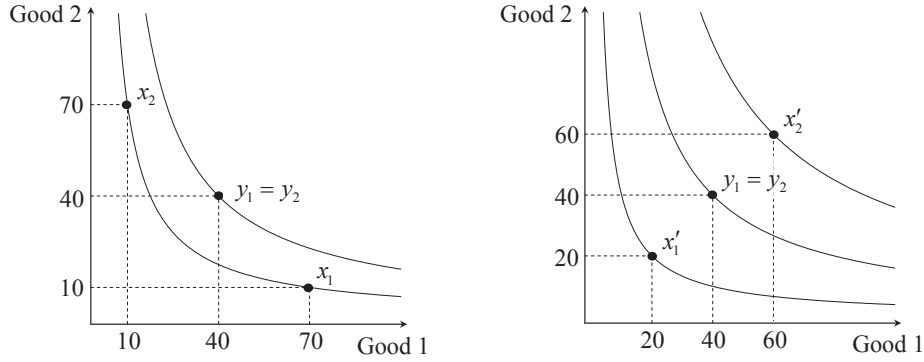


Figure 1. Distribution  $X$  exhibits inefficiency but is perfectly equitable (left), while distribution  $X'$  exhibits inequity but is perfectly efficient (right)

available goods in  $X$ . In distribution  $X$ , the two individuals have equal well-being levels as measured by the symmetric Cobb-Douglas function, and hence  $X$  is perfectly equitable. The entire welfare gain in moving to the ideal distribution  $Y$  is due to efficiency improving exchanges of goods between the individuals. Consider now the right-hand panel, which depicts distribution  $X' = (x'_1, x'_2)$ , where  $x'_1 = (20, 20)$  and  $x'_2 = (60, 60)$ . In distribution  $X'$ , there are no possibilities for efficiency improving exchanges of goods, and hence  $X'$  is perfectly efficient.<sup>4</sup> Now the entire welfare gain in moving to the ideal distribution  $Y$  is due to equity improving transfers of goods from the better off individual 2 to the worse off individual 1.

The example shows that the sources of potential welfare gain, and hence the meaning of inequality in the normative approach, can differ dramatically depending on the distribution under consideration. Indeed, inequality coincides with inefficiency in the case of distribution  $X$ , while it coincides with inequity in the case of distribution  $X'$ . Incidentally, the inequality values computed using the assumed double-Cobb-Douglas social welfare specification are 0.34 for  $X$  and 0.13 for  $X'$ .<sup>5</sup> This illustrates that it is highly misleading to identify inequality with inequity in the normative approach: although measured inequality is greater in  $X$  than in  $X'$ , inequity is smaller in  $X$  than in  $X'$ .

Distributions typically exhibit both inequity and inefficiency. A full un-

<sup>4</sup>Indeed, the marginal rates of substitution of the two individuals are equal.

<sup>5</sup>The inequality measure used is  $1 - f$ , where  $f$  is obtained by substituting equations (5) and (6) into equation (2) and setting  $\alpha = 1$ ,  $\beta = 1$  and  $r_1 = r_2$ .

derstanding of multidimensional inequality therefore requires distinguishing and quantifying these two underlying aspects. Using a suggestion by Graaff (1977), we show that normative indices of multidimensional inequality can be neatly decomposed into an inequity and an inefficiency part. We provide an exact formula for each of the two components for a generic class of social welfare functions. This generic class includes the popular double-CES class, which in turn includes the double-Cobb-Douglas social welfare function used in the example above.

One striking conclusion of our analysis is that the inequity component associated with a normative inequality measure is a so-called two-stage measure, as proposed by Maasoumi (1986). Such a measure applies a unidimensional inequality measure to the vector of individual well-being levels. The two-stage approach has been regarded as both intrinsically different from the normative approach and conceptually less appealing (Bourguignon, 1999, and Weymark, 2006). But the analysis shows that the two-stage approach receives a solid theoretical justification as a measure of multidimensional inequity from within the normative approach.

We use the decomposition to assess two prominent multidimensional transfer principles, viz., uniform majorization and correlation increasing majorization, both of which have received criticism in the literature. First, Dardanoni (1995) has argued that uniform majorization in some cases propagates transfers that increase the gaps between individual well-being levels. Our decomposition shows that the transfers in such cases indeed worsen equity, but that they improve efficiency, so that uniform majorization can be seen as demanding that the latter effect outweighs the former effect. Second, Bourguignon and Chakravarty (2003) have argued that correlation increasing majorization is inappropriate if the goods are complements rather than substitutes. Our decomposition suggests that this point may be stated too harshly. Given complementarity, the transfers propagated by the principle indeed cause an efficiency loss. But because there is also an equity improvement, the principle can be understood as demanding tolerance of this efficiency leak in return for the improvement in equity.

Finally, we illustrate the proposed decomposition with an empirical application to population-weighted between-country inequality in the period 1980 to 2011. We focus on the three well-being dimensions of the popular Human Development Index: living standards, health and education. We show that multidimensional inequality has decreased during the period and that this result is driven by decreases in both inequity and inefficiency.

The next section introduces notation and basic assumptions. Section

3 presents the decomposition of multidimensional inequality into inequity and inefficiency parts. In Section 4, uniform majorization and correlation increasing majorization are critically discussed in the light of the decomposition. Section 5 provides the illustrative empirical analysis. Section 6 concludes.

## 2 Notation and assumptions

There are  $n$  individuals and  $m$  goods. The quantity of good  $k$  owned by individual  $i$  is a positive real number  $x_{ik}$ . The bundle of individual  $i$  is a vector  $x_i = (x_{i1}, x_{i2}, \dots, x_{im})$ . A distribution is an  $n \times m$  matrix  $X$  with bundle  $x_i$  at the  $i$ th row. The set of all bundles is  $B$  and the set of all distributions is  $D$ . We use  $\mu_X$  to denote the bundle  $(\mu_1, \mu_2, \dots, \mu_m)$  with  $\mu_k = (x_{1k} + x_{2k} + \dots + x_{nk})/n$  the average quantity of good  $k$  in distribution  $X$ . We use  $X_\mu$  to denote the distribution in which each individual has a bundle equal to  $\mu_X$ . We write  $1_\ell$  for the  $\ell$ -vector with a one at each entry.

A social welfare function is a function  $W : D \rightarrow \mathbb{R}$ . The value  $W(X)$  is to be interpreted as the social welfare level associated with distribution  $X$  in  $D$ . We focus on the class of social welfare functions  $\mathcal{W}$ . A social welfare function  $W$  is in  $\mathcal{W}$  if and only if there exist a continuous, concave, linearly homogenous<sup>6</sup> and strictly increasing function  $u : B \rightarrow \mathbb{R}$  with  $u(1_m) = 1$  and a continuous, Schur-concave,<sup>7</sup> linearly homogenous and strictly increasing function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $v(1_n) = 1$  such that, for each distribution  $X$  in  $D$ , we have

$$W(X) = v(u(x_1), u(x_2), \dots, u(x_n)). \quad (1)$$

As we prove in Appendix A, the class  $\mathcal{W}$  is characterized by the following seven properties: (i) anonymity (welfare is invariant to rearranging bundles of individuals), (ii) monotonicity (increasing the amount of a good owned by an individual increases welfare), (iii) continuity (small changes in distributions do not cause large changes in their welfare ranking), (iv) weak uniform majorization (progressive transfers uniformly applied to each good do not decrease welfare), (v) normalization (if each individual has an amount  $\lambda$  of each good, then the social welfare level is  $\lambda$ ), (vi) individualism (welfare is measured in two steps, a first step to aggregate across

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<sup>6</sup>A function  $\psi : \mathbb{R}^\ell \rightarrow \mathbb{R}$  is linearly homogenous, or homogenous of degree one, if  $\psi(\lambda s) = \lambda \psi(s)$  for each  $s$  in  $\mathbb{R}^\ell$  and each positive real number  $\lambda$ .

<sup>7</sup>A function  $\psi : \mathbb{R}^\ell \rightarrow \mathbb{R}$  is Schur-concave if  $\psi(s) \leq \psi(Qs)$  for each  $s$  in  $\mathbb{R}^\ell$  and each bistochastic matrix  $Q$ . A bistochastic matrix is a nonnegative square matrix of which each row sum and each column sum is equal to 1. A permutation matrix is a bistochastic matrix of which each component is either 0 or 1.

dimensions for each individual, and a second step to aggregate the obtained values across individuals) and (vii) homotheticity (the welfare ranking of two distributions is preserved if each good of each individual is multiplied by the same factor in both distributions). These seven properties are standard in the literature.<sup>8</sup> Nevertheless, the use of weak uniform majorization is not without controversy, an issue to which we return in detail in Section 4.

We will interpret the function  $u$  in equation (1) as a well-being measure common to all individuals. This interpretation is in line with the literature. Some studies explicitly treat  $u$  (or a similar function) as a measure of individual well-being and discuss it using the language of utility theory (e.g., Atkinson and Bourguignon, 1982, Atkinson, 2003, and Bourguignon and Chakravarty, 2003). Other studies are less explicit, but admit the possibility of this interpretation. For example, Tsui (1995, p. 264) notes that properties such as (i) to (vii) above do not only impose requirements on a social welfare function, but also on an underlying concept of individual well-being.<sup>9</sup> Hence, these properties can be interpreted as describing an ethical observer's value judgments concerning individual well-being, a possible interpretation already noted by Kolm (1977, pp. 2-3).

Two final comments are in order. First, the function  $u$  is a least concave representation of the common individual well-being ranking (see Proposition 1 in Kihlstrom and Mirman, 1981).<sup>10</sup> Therefore, the social welfare function in equation (1) is in a form that allows a clear separation between, on the one hand, the ordinal properties of the individual well-being ranking, such as the degree of complementarity of the goods and, on the other hand, the cardinal properties of the social aggregation, such as the

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<sup>8</sup>With the exception of normalization and individualism, all these properties are discussed in the survey by Weymark (2006). Normalization is an innocent property that ensures a convenient cardinalization of the social welfare function. Individualism is very common in the literature: some studies explicitly impose individualism (e.g., Kolm, 1977, and Seth, 2011), while others assume stronger requirements of additive separability that imply individualism (e.g., Tsui, 1995). A noteworthy exception is the study by Gajdos and Weymark (2005). Gajdos and Weymark propose a social welfare function that is not individualistic, but rather first aggregates across individuals for each dimension, and second aggregates the obtained values across dimensions. Their approach can be seen as an attempt to make the best of a situation in which no data on the joint distribution of the dimensions are available.

<sup>9</sup>For example, anonymity makes the individual well-being measure common to all individuals, monotonicity makes it increasing in the amounts of goods, and weak uniform majorization makes it quasi-concave.

<sup>10</sup>A function  $v$  is a least concave representation of a well-being relation if, for each concave representation  $v^*$  of the same well-being relation, we have  $v^* = \psi \circ v$ , where  $\psi$  is a concave and increasing function (Debreu, 1976).

degree of inequity aversion. With the cardinal representation of the individual well-being ranking fixed, the former category of properties is in  $u$ , whereas the latter category is in  $v$ . This form was advocated by Atkinson and Bourguignon (1982, p. 191).

Second, the value  $W(X)$  in equation (1) is the equally distributed equivalent level of individual well-being associated with  $X$ . That is,  $W(X)$  is the level of individual well-being that, if equally attained by all individuals, yields the same level of welfare as  $X$ . The function  $W$  inherits linear homogeneity from the function  $u$ .

### 3 Equity and efficiency as components of equality

#### 3.1 Measuring inequality

The normative approach to inequality measurement identifies inequality with the potential welfare gain that could be obtained by distributing the available goods optimally. In order to avoid that the values of the inequality measure depend on an arbitrary choice of cardinalization of the social welfare function, the literature has opted to measure this potential welfare gain in terms of goods rather than directly in terms of welfare. Kolm (1977, footnote 3) and Tsui (1995) proposed what is now the standard procedure to derive a relative multidimensional inequality measure from a social welfare function in  $\mathcal{W}$ . A relative inequality measure is invariant to a multiplication of each good of each individual by the same factor. Appendix B provides the analogous analysis for absolute inequality measures (which are invariant to an addition of the same amount to each good of each individual).

The Kolm-Tsui procedure is as follows: equality in distribution  $X$  is measured by the smallest fraction of the total good amounts in  $X$  required to maintain the level of welfare of  $X$ .

*Definition 1.* Let  $X$  be a distribution in  $D$ . Let  $f(X)$  be the smallest real number for which there exists a distribution  $Y$  in  $D$  such that  $W(Y) = W(X)$  and  $\mu_Y = f(X) \times \mu_X$ . Then,  $f(X)$  is the equality level and  $1 - f(X)$  is the inequality level of distribution  $X$ .

Because a social welfare function  $W$  in  $\mathcal{W}$  satisfies weak uniform majorization, the available goods in  $X$  are optimally distributed if each individual is given the average bundle  $\mu_X$ . Hence, equality  $f(X)$  is defined by  $W(X) = W(f(X)X_\mu)$ . Since  $W$  is linearly homogenous, we have

$$f(X) = \frac{W(X)}{W(X_\mu)}. \quad (2)$$



The relative inequality measure  $1 - f$  is standard in the literature.<sup>11</sup> The inequality level  $1 - f(X)$  is the percentage welfare loss incurred by moving from the optimal distribution  $X_\mu$  to the actual distribution  $X$ .

The example in the introduction revealed that the potential welfare gain measured by  $1 - f$  reflects both efficiency and equity improvements. This makes the meaning and interpretation of variations in inequality ambiguous, thus calling for a decomposition of equality into efficiency and equity parts. Not only will such a decomposition illuminate existing controversies surrounding the theoretical foundations of multidimensional inequality measurement (see Section 4), it will also be a useful tool in empirical research (see Section 5).

### 3.2 A decomposition

To decompose equality into efficiency and equity parts, we use a procedure suggested by Graaff (1977).<sup>12</sup> Graaff's procedure distinguishes two steps involved in the welfare optimization underlying Definition 1. The first step consists of efficiency improving exchanges of goods, which increase the well-being levels of all involved individuals. The second step consists of equity improving redistributions of goods, which decrease the individual well-being gaps. Graaff proposes an efficiency measure and an equity measure corresponding, respectively, to each of these two steps.

The efficiency measure is closely related to Debreu's (1951) coefficient of resource utilization: efficiency in distribution  $X$  is measured by the smallest fraction of the total good amounts in  $X$  required to maintain the individual well-being levels associated with  $X$ .

*Definition 2.* Let  $X$  be a distribution in  $D$ . Let  $d(X)$  be the smallest real number for which there exists a distribution  $Z$  in  $D$  such that  $u(z_i) = u(x_i)$  for each individual  $i$  and  $\mu_Z = d(X) \times \mu_X$ . Then,  $d(X)$  is the efficiency level and  $1 - d(X)$  is the inefficiency level of distribution  $X$ .

Distribution  $Z$  in Definition 2 is perfectly efficient, but it maintains the level of inequity of distribution  $X$  because all individual well-being levels are the same in the two distributions. Therefore, the level of equity of distribution  $X$  can be measured by applying Definition 1 to distribution

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<sup>11</sup>The linear homogeneity of  $W$  indeed ensures that  $1 - f$  is relative, i.e., we have  $1 - f(X) = 1 - f(\lambda X)$  for each distribution  $X$  in  $D$  and each positive real number  $\lambda$ .

<sup>12</sup>Graaff (1977) did not apply his procedure to multidimensional inequality measurement. Graaff was instead interested in establishing the conditions under which aggregate income growth leads to an improvement in efficiency.

$Z$ : equity in distribution  $X$  is the smallest fraction of the good amounts in distribution  $Z$  required to maintain the level of welfare of  $Z$ .

*Definition 3.* Let  $X$  be a distribution in  $D$ . Let  $Z$  be a distribution in  $D$  obtained from distribution  $X$  as in Definition 2. Let  $e(X)$  be the smallest real number for which there exists a distribution  $T$  such that  $W(T) = W(Z)$  and  $\mu_T = e(X) \times \mu_Z$ . Then,  $e(X)$  is the equity level and  $1 - e(X)$  is the inequity level of distribution  $X$ .

As in Graaff (1977), we obtain a multiplicative decomposition of equality into its efficiency and equity parts. To see this, first note that  $W(X) = W(Z)$  because all well-being levels are equal in distributions  $X$  and in  $Z$ . Hence, distribution  $T$  in Definition 3 must be equal to distribution  $Y$  in Definition 1. Combining  $\mu_Y = e(X) \times \mu_Z$  and  $\mu_Z = d(X) \times \mu_X$ , we have  $\mu_Y = e(X) \times d(X) \times \mu_X$ . Since also  $\mu_Y = f(X) \times \mu_X$ , we indeed obtain

$$f(X) = d(X) \times e(X).$$

We will now derive formulas for the efficiency measure  $d$  and the equity measure  $e$ . We start by constructing distribution  $Z$  in Definition 2 for a given distribution  $X$ . Figure 2 gives an illustration for the two-individual case. Because the individual well-being function  $u$  is homothetic, giving each individual a proportion of the same bundle  $n\mu_X$  results in an efficient distribution.<sup>13</sup> For each individual  $i$ , define  $s_i$  by  $u(x_i) = u(s_i n\mu_X)$ . That is,  $s_i$  is the share of the societal bundle  $n\mu_X$  that individual  $i$  would require in order to attain the level of well-being associated with bundle  $x_i$ . Hence, distribution  $Z$  equals  $(s_1 n\mu_X, s_2 n\mu_X, \dots, s_n n\mu_X)'$ . Efficiency  $d(X)$  is computed by the relative distance between the actual societal bundle  $n\mu_X$  and the smaller bundle  $n\mu_Z$ . Since  $n\mu_Z = \sum_{i=1}^n s_i n\mu_X$ , we have  $d(X) = \sum_{i=1}^n s_i$ . Moreover,  $s_i = u(x_i)/nu(\mu_X)$  because  $u$  is linearly homogenous. We obtain

$$d(X) = \frac{\frac{1}{n} \sum_{i=1}^n u(x_i)}{u(\mu_X)}. \quad (3)$$

Equation (3) has a straightforward interpretation. It reveals that the efficiency measure  $d$  is in fact the equality measure  $f$  in equation (2) applied to the inequity neutral version of the social welfare function  $W$ , i.e., the social welfare function obtained by replacing in equation (1) the function  $v$  by the mean operator.<sup>14</sup>

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<sup>13</sup>To see this, note that the marginal rates of substitution (assuming these are well-defined) of the individuals are equal if all bundles are on the same ray through the

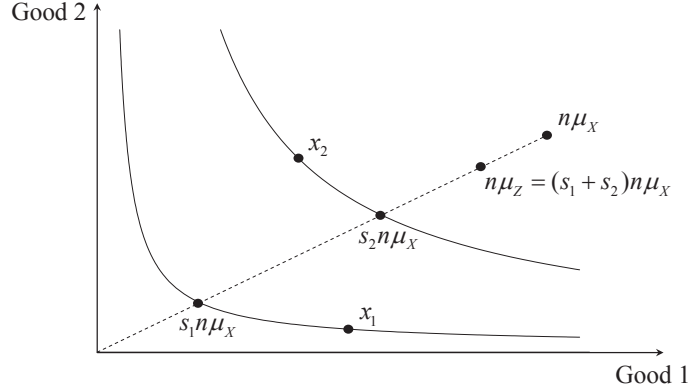


Figure 2. The efficiency level of distribution  $X$  is  $d(X) = s_1 + s_2$

To provide a formula for  $e$ , we use that  $e = f/d$ . Substituting equations (2) and (3) and using that  $u(\mu_X) = W(X_\mu)$  for each social welfare function  $W$  in  $\mathcal{W}$ , we obtain

$$e(X) = \frac{W(X)}{\frac{1}{n} \sum_{i=1}^n u(x_i)}. \quad (4)$$

Equation (4) is the equally distributed equivalent well-being level divided by the average well-being level, so that the inequality measure  $1 - e$  coincides with Sen's (1973, p. 42, equation (2.17)) general normative relative unidimensional inequality measure applied to the vector of well-being levels. Hence, the inequality measure  $1 - e$  is what Weymark (2006) refers to as a *two-stage* inequality measure. Indeed, in the two-stage approach, which was proposed by Maasoumi (1986), the individual well-being levels are computed in the first stage and a unidimensional inequality measure is applied to the vector of these levels in the second stage.<sup>15</sup> The two-stage approach has been interpreted in the literature as being inherently distinct from the normative approach as well as theoretically less appealing (see the discussions by Bourguignon, 1999, and Weymark, 2006). Hence, it

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origin.

<sup>14</sup>Inequity aversion is defined here completely in analogy with risk aversion in the case of many commodities (Kihlstrom and Mirman, 1974). The fact that the mean of least concave representations can be interpreted as the risk neutral, or inequity neutral, case is discussed by Kihlstrom and Mirman (1981).

<sup>15</sup>Maasoumi (1986) proposed to apply the generalized entropy class of unidimensional inequality measures to the vector of levels of a CES individual well-being measure. The generalized entropy class includes a subclass that is ordinally equivalent to the Atkinson (1970) class of normative unidimensional inequality measures.

is striking that the two-stage approach turns out to deliver the inequity component of the normative inequality measure  $1 - f$ .

Note that the inefficiency measure and the inequity measure have a straightforward graphical interpretation in terms of Figure 2. The inefficiency measure summarizes the distance from the actual bundles to the constructed efficient bundles on the dashed line. The inequity measure measures the inequality between these constructed efficient bundles on the dashed line.<sup>16</sup>

### 3.3 The double-CES class

We end this section with an application of the decomposition to the class of double-CES social welfare functions, a popular subclass of  $\mathcal{W}$  of which the members satisfy additive separability and replication invariance.<sup>17</sup> The double-CES class is used in the empirical illustration of Section 5.

A double-CES social welfare function uses a CES function to aggregate at the social level and a second CES function to aggregate at the individual level. A social welfare function  $W$  is a member of the double-CES class if, for each distribution  $X$  in  $D$ , we have

$$W(X) = \begin{cases} \left( \frac{1}{n} \sum_{i=1}^n u(x_i)^{1-\alpha} \right)^{\frac{1}{1-\alpha}} & \text{for } \alpha \geq 0 \text{ and } \alpha \neq 1, \\ \prod_{i=1}^n u(x_i)^{\frac{1}{n}} & \text{for } \alpha = 1, \end{cases} \quad (5)$$

and

$$u(x_i) = \begin{cases} \left( \sum_{k=1}^m r_k x_{ik}^{1-\beta} \right)^{\frac{1}{1-\beta}} & \text{for } \beta \geq 0 \text{ and } \beta \neq 1, \\ \prod_{k=1}^m x_{ik}^{r_k} & \text{for } \beta = 1, \end{cases} \quad (6)$$

where  $r_k > 0$  for each good  $k$  and  $r_1 + r_2 + \dots + r_m = 1$ . The shape of the iso-well-being curves is determined by the weights  $r_k$  on the different dimensions and by the parameter  $\beta$ . The parameter  $\beta$  determines the

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<sup>16</sup>Because  $1 - e$  is a relative two-stage measure, it is simply a unidimensional measure applied to the  $s_i$  shares.

<sup>17</sup>For axiomatic characterizations of the double-CES class (or subclasses thereof), see Decancq and Ooghe (2010), Lasso de la Vega and Urrutia (2011), Seth (2011) and Tsui (1995). For discussions of the class, see Atkinson (2003) and Bourguignon (1999).

degree of complementarity, with  $\beta = 0$  corresponding to the case of perfect substitutes and  $\beta \rightarrow \infty$  to the case of perfect complements. The parameter  $\alpha$  determines the degree of inequity aversion. The case  $\alpha = 0$  corresponds to inequity neutrality with social welfare equal to the mean of least concave representations, and the case  $\alpha \rightarrow \infty$  corresponds to maximin with welfare measured by the well-being level of the individual in the worst position.

We apply equations (2), (3) and (4) to the double-CES class (we omit the cases  $\alpha = 1$  and  $\beta = 1$ ). We have

$$f(X) = \left[ \frac{1}{n} \sum_{i=1}^n \left( \frac{\sum_{k=1}^m r_k x_{ik}^{1-\beta}}{\sum_{k=1}^m r_k \mu_k^{1-\beta}} \right)^{\frac{1-\alpha}{1-\beta}} \right]^{\frac{1}{1-\alpha}}, \quad (7)$$

$$d(X) = \frac{1}{n} \sum_{i=1}^n \left( \frac{\sum_{k=1}^m r_k x_{ik}^{1-\beta}}{\sum_{k=1}^m r_k \mu_k^{1-\beta}} \right)^{\frac{1}{1-\beta}}, \quad (8)$$

and

$$e(X) = \frac{\left( \frac{1}{n} \sum_{i=1}^n \left( \sum_{k=1}^m r_k x_{ik}^{1-\beta} \right)^{\frac{1-\alpha}{1-\beta}} \right)^{\frac{1}{1-\alpha}}}{\frac{1}{n} \sum_{i=1}^n \left( \sum_{k=1}^m r_k x_{ik}^{1-\beta} \right)^{\frac{1}{1-\beta}}}. \quad (9)$$

The parameters  $\alpha$  and  $\beta$  of the double-CES class play different roles for the inefficiency and inequity components.<sup>18</sup> First, the interpretation of  $\alpha$  as a parameter of inequity aversion is clear: efficiency  $d$  is not influenced by  $\alpha$ , while inequity  $1-e$  increases if inequity aversion  $\alpha$  increases. Hence, also the relative importance of the inequity component increases with inequity aversion. Inequality  $1-f$  is a pure measure of inefficiency if  $\alpha = 0$ , i.e., in that case  $1-f = 1-d$ . Second, as the parameter  $\beta$  increases, the different dimensions become less substitutable and efficiency  $d$  lowers. Typically, also the value of  $e$  will change as  $\beta$  varies. The reason is that  $1-e$  is equal to the unidimensional Atkinson (1970) inequality measure applied to the vector of well-being levels, and these well-being levels typically change if  $\beta$  changes. Inequality  $1-f$  reduces to a pure measure of inequity only if the goods are perfect substitutes, i.e., if  $\beta = 0$ . In this case we have  $1-f = 1-e$ .<sup>19</sup>

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<sup>18</sup>The values of the weights  $r_k$  for the different goods have a complex impact on both  $d$  and  $e$ .

<sup>19</sup>For a characterization of a multidimensional social welfare function with a linear individual well-being function, see Bosmans, Lauwers and Ooghe (2009).

## 4 How do transfers affect equity and efficiency?

Multidimensional transfer principles endow the social welfare function with a concern for equality. They therefore play a key role in the normative approach to inequality measurement. Two prominent transfer principles are uniform majorization (Kolm, 1977, Tsui, 1995) and correlation increasing majorization (Atkinson and Bourguignon, 1982, Tsui, 1999). Although these two principles are widely used in the literature, both have received considerable criticism. Uniform majorization has been shown to recommend in some cases transfers that are unambiguously unfavorable in terms of equity (Dardanoni, 1995). Correlation increasing majorization has been claimed to be inappropriate if the goods are complements (Bourguignon and Chakravarty, 2003). We will disentangle the efficiency and equity effects of the transfers propagated by each of the two principles and thus provide new insights into these two critiques.

First, we consider uniform majorization, which advocates progressive transfers in each dimension with a uniform structure across dimensions. The principle is a strict version of the weak uniform majorization principle defined in Section 2.

*Uniform majorization.* For all distributions  $X$  and  $Y$  in  $D$  such that  $X \neq Y$ , if  $X = QY$  with  $Q$  a bistochastic matrix that is not a permutation matrix, then  $W(X) > W(Y)$ .

Uniform majorization is satisfied by a social welfare function in  $\mathcal{W}$  if  $u$  is strictly concave and  $v$  is strictly Schur-concave.

Dardanoni (1995) provides an example in which uniform majorization recommends transfers that actually widen the gaps between individual well-being levels. He argues that this puts into question the appropriateness of uniform majorization in the context of inequality measurement. Figure 3 presents an example qualitatively similar to Dardanoni's. The figure extends the example in the left-hand panel of Figure 1 by adding in distributions  $X$  and  $Y$  a third individual with bundle  $x_3 = y_3 = (10, 10)$ .<sup>20</sup> Equity clearly worsens in going from  $X$  to  $Y$  because the gap between the equally well off individuals 1 and 2, on the one hand, and the worst off individual 3, on the other hand, further widens. Nevertheless, uniform majorization demands that equality increases as we move from  $X$  to  $Y$ , i.e.,  $f(X) < f(Y)$ .<sup>21</sup> Our decomposition helps to understand this coun-

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<sup>20</sup>The iso-well-being curves in the figure are for the symmetric Cobb-Douglas case. This is not essential: examples with the same conclusion can easily be constructed if  $u$  is not in the Cobb-Douglas form or is not symmetric.

<sup>21</sup>Because  $Y = QX$  and  $X \neq Y$  with  $Q = ((0.5, 0.5, 0), (0.5, 0.5, 0), (0, 0, 1))$ , we have

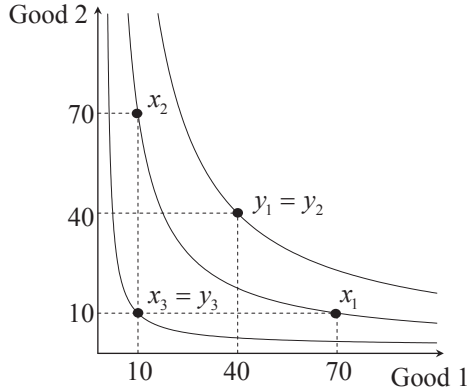


Figure 3. Uniform majorization implies  $W(X) < W(Y)$

terintuitive implication. It is straightforward to show that, for all social welfare functions in  $\mathcal{W}$  satisfying uniform majorization, the move from  $X$  to  $Y$  is a worsening in terms of equity, but an improvement in terms of efficiency, i.e., we have  $e(X) > e(Y)$  and  $d(X) < d(Y) = 1$ . That is, uniform majorization imposes that the efficiency improvement in going from  $X$  to  $Y$  should outweigh the equity worsening, so that equality as measured by  $f$  goes up.<sup>22</sup>

The conclusion that the transfers recommended by uniform majorization increase efficiency extends beyond the above example. It holds in general: if  $u$  is strictly concave, then we have  $d(QX) > d(X)$  for each distribution  $X$  in  $D$  and each non-permutation bistochastic matrix  $Q$  such that  $X \neq QX$ .<sup>23</sup> In terms of Figure 2, the intuition is that multiplication by a bistochastic matrix brings the bundles closer to the dashed line. To sum up, the effect on efficiency is positive in general, while, as the above example shows, the effect on equity is not always positive. We can con-

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$W(X) < W(Y)$ . Since, in addition,  $W(X_\mu) = W(Y_\mu)$ , we obtain  $f(X) < f(Y)$  using equation (2).

<sup>22</sup>Dardanoni (1995) uses a two-stage inequality measure to argue that ‘inequality’ increases in this example. Weymark (2006) argues that Dardanoni’s example shows that it is the two-stage approach that is problematical, rather than the uniform majorization principle. But, we saw in the previous section that the inequity measure in the normative approach is a two-stage inequality measure. Hence, if we interpret Dardanoni’s critique as arguing that uniform majorization is not a satisfactory principle of equity, then his critique remains valid also within the normative approach.

<sup>23</sup>Let  $Y = QX$ . We have  $\sum_{i=1}^n u(x_i)/n < \sum_{i=1}^n u(y_i)/n$  (Kolm, 1977, Theorem 3), while  $u(\mu_X) = u(\mu_Y)$ . Using equation (3), we obtain  $d(QX) > d(X)$ .

clude, therefore, that uniform majorization is more successful at capturing the efficiency aspect of multidimensional inequality than at capturing the equity aspect.<sup>24</sup>

Next, we discuss correlation increasing majorization, which advocates exchanges of goods that decrease the correlation between dimensions. Let  $X$  and  $Y$  be distributions in  $D$  that coincide except for individuals  $i$  and  $j$ , let  $x_i \neq x_j$ , and let  $y_i$  be the componentwise minimum and  $y_j$  the componentwise maximum of  $x_i$  and  $x_j$ .<sup>25</sup> Then,  $Y$  is said to be obtained from  $X$  by a correlation increasing switch.

*Correlation increasing majorization.* For all distributions  $X$  and  $Y$  in  $D$ , if  $Y$  is obtained from  $X$  by a correlation increasing switch, then  $W(X) > W(Y)$ .

Figure 4 gives an example where distribution  $X = (x_1, x_2)$  with  $x_1 = (70, 10)$  and  $x_2 = (10, 70)$  and distribution  $Y = (y_1, y_2)$  with  $y_1 = (70, 70)$  and  $y_2 = (10, 10)$ . Conditions on  $u$  and  $v$  for a social welfare function in  $\mathcal{W}$  that ensure satisfaction of correlation increasing majorization have not been established in the literature. For a member of the double-CES class, the requirement is  $\alpha > \beta$ .

Bourguignon and Chakravarty (2003, pp. 35-36) claim that correlation increasing majorization is inappropriate if the goods are complements. We use our decomposition to examine this claim. To define complementarity we rely on the Auspitz-Lieben-Edgeworth-Pareto (ALEP) notion as refined by Kannai (1980): goods are complements if  $u$  is strictly supermodular.<sup>26</sup>

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<sup>24</sup>It is by now fair to say that the uniform majorization principle is controversial as a principle of equity. Trannoy (2006) and Fleurbaey (2006) argue that it is inappropriate that the principle imposes quasi-concavity on individual well-being functions. Duclos, Sahn and Younger (2011) show that an example where the principle recommends transfers that reduce equity can also be found in the two-individual case if the assumption of homothetic individual well-being rankings is dropped.

<sup>25</sup>That is,  $y_i = (\min\{x_{ik}, x_{jk}\})_{k=1,\dots,m}$  and  $y_j = (\max\{x_{ik}, x_{jk}\})_{k=1,\dots,m}$ .

<sup>26</sup>A function  $u$  is supermodular if  $u(x \vee y) + u(x \wedge y) \geq u(x) + u(y)$  for all  $x$  and  $y$  in  $B$ , where  $x \vee y$  is the componentwise maximum and  $x \wedge y$  is the componentwise minimum of  $x$  and  $y$ . If, in addition, the inequality holds strictly for all unordered pairs  $x$  and  $y$ , then  $u$  is strictly supermodular. For a twice-differentiable  $u$ , supermodularity is equivalent to non-negative cross-derivatives. According to the ALEP notion, complementarity is equivalent to strict supermodularity of some functional representation of the well-being ranking. The refinement of Kannai (1980) consists in applying the ALEP notion to the specific least concave representation of the well-being ranking and thus removes all arbitrariness concerning the choice of the representation. Unfortunately, Bourguignon and Chakravarty (2003) rely on the original ALEP definition, which renders their treatment of complements versus substitutes unsatisfactory. We refer to Atkinson (2003, pp. 57-60) for a thorough treatment of this point.



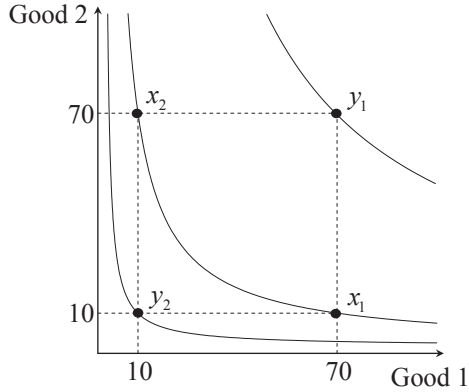


Figure 4. Correlation increasing majorization implies  $W(X) > W(Y)$

A general conclusion can be drawn about the effect of a correlation increasing switch on efficiency: if  $u$  is strictly supermodular, then we have  $d(X) < d(Y)$  for all distributions  $X$  and  $Y$  in  $D$  such that  $Y$  is obtained from  $X$  by a correlation increasing switch.<sup>27</sup> In other words, the transfers advocated by correlation increasing majorization always worsen efficiency in the case of complements. In terms of Figure 2, the intuition is roughly that these transfers move the bundles away from the dashed line. But the transfers also unambiguously reduce the well-being gap between the two individuals involved, irrespective of whether the goods are complements or substitutes. Summing up, correlation increasing majorization demands we accept an efficiency leak in a transfer from a better off individual to a worse off individual. Hence, it does not follow that correlation increasing majorization is a priori inappropriate if the goods are complements: the principle just becomes more demanding, i.e., imposes a greater tolerance of efficiency leaks, as the degree of complementarity increases.

We consider the distributions  $X$  and  $Y$  in Figure 4 to make the above point more concrete. For a strictly supermodular  $u$ , we indeed have  $d(X) < d(Y) = 1$ , while the gap between the well-being levels of individuals 1 and 2 has widened in going from  $X$  to  $Y$ . The efficiency loss that has to be tolerated in return for the reduction in the well-being gap can be made precise by considering the well-being vectors corresponding to distributions  $X$

<sup>27</sup>Let  $Y$  be obtained from  $X$  by a correlation increasing switch. We have  $\sum_{i=1}^n u(x_i)/n < \sum_{i=1}^n u(y_i)/n$  by supermodularity of  $u$ , while  $u(\mu_X) = u(\mu_Y)$ . Using equation (3), we obtain  $d(X) < d(Y)$ .

and  $Y$ . In Figure 4 we have  $\beta = 1$  and hence  $u$  is strictly supermodular. We have  $(u(x_1), u(x_2)) = (27, 27)$  and  $(u(y_1), u(y_2)) = (70, 10)$ . The efficiency loss in moving from  $Y$  to  $X$  translates in terms of well-being as a leaky-bucket transfer: individual 1 gave up 43 units, but individual 2 received only 17. The leak that the principle asks us to tolerate increases as the degree of complementarity  $\beta$  increases. For example, this leak is zero units if the goods are substitutes ( $\beta = 0$ ), it is 12 units if  $\beta = 0.5$ , it is 26 units if  $\beta = 1$  as we saw above, and it approaches the full 60 unit well-being difference between individuals 1 and 2 as  $\beta \rightarrow \infty$ . This also helps to understand the requirement  $\alpha > \beta$  for a double-CES social welfare function: as the degree of complementarity  $\beta$  increases, the efficiency leak becomes larger, and hence the tolerance of leaks has to increase accordingly by increasing inequity aversion  $\alpha$ .

The analysis of this section shows that neither uniform majorization nor correlation increasing majorization captures the simplicity of the unidimensional Pigou-Dalton principle. The Pigou-Dalton principle expresses a clear and elementary idea: an improvement in equity (in the unidimensional case, a progressive transfer), while keeping efficiency at the same level (which in the unidimensional case amounts to preserving the mean), increases welfare. On the other hand, we have seen that the two multidimensional principles have implications in cases where equity and efficiency change simultaneously. In the case of uniform majorization the recommendation sometimes even is unequivocally in the direction of worse equity. We conclude that it would be interesting to examine a new multidimensional transfer principle that, like the unidimensional Pigou-Dalton principle, recommends transfers that improve equity  $e$  but preserve efficiency  $d$ . We leave this issue to future research.<sup>28</sup>

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<sup>28</sup>For a social welfare function in  $\mathcal{W}$ , such efficiency-preserving transfers would boil down to simple progressive transfers in the individual well-being vector, i.e., transfers that preserve  $\sum_{i=1}^n u(x_i)$  (and  $\mu(X)$ ): see equation (3). Hence, such a transfer principle is trivially satisfied for each social welfare function in  $\mathcal{W}$  with a strictly Schur-concave  $v$ . But there is a circular idea here because the social welfare functions in  $\mathcal{W}$  already satisfy weak uniform majorization, a property we may no longer want once we impose the new transfer principle. Moreover, weak uniform majorization was used to obtain the formula for efficiency in equation (3) in the first place. Bosmans, Decancq and Ooghe (2013) study an efficiency-preserving transfer principle in the more general context of heterogenous individual well-being rankings.

## 5 Empirical illustration

We illustrate the proposed decomposition with an empirical application to population-weighted between-country inequality in the period 1980 to 2011. Each individual in a country is assigned the same bundle of goods, the average bundle of his country. Following the well-known Human Development Index (HDI), we consider the following three ‘goods’: living standards, health and education.<sup>29</sup> We only retain the countries for which the data are available in both 1980 and 2011, leaving us with 105 countries covering 84% of the population in 2011. Appendix C lists the countries. We measure inequality and its inefficiency and inequity components using the double-CES class in equations (7), (8) and (9). Throughout, we use equal weights for the three goods, i.e.,  $r_1 = r_2 = r_3 = 1/3$ , but we consider a variety of values of  $\alpha$  and  $\beta$ .

The HDI aggregates the goods in a way that is consistent with our analysis. It uses the CES well-being function in equation (3) with equal weighting of the three goods and with  $\beta$  equal to 0 prior to 2010 and  $\beta$  equal to 1 since. By consequence, the inequity component in our analysis is the unidimensional Atkinson inequality measure applied to the HDI well-being measure (for  $\beta$  equal to 0 or 1) or to similar well-being measures (for other values of  $\beta$ ).<sup>30</sup> The recent change in the value of  $\beta$  in the definition of the HDI is interesting, as it constitutes a move away from perfect substitutability ( $\beta = 0$ ) to a case of complementarity ( $\beta = 1$ ), and hence introduces the efficiency aspect. In the current context, efficiency captures, using the words of the UNDP (2010, p. 15), “how well rounded a country’s performance is across the three dimensions.”

Figure 5 presents an overview of the data. The two left-hand panels are for 1980, the two right-hand panels for 2011. The two top panels show each country’s achievements in living standards and health, the two bottom panels show achievements in living standards and education. Larger

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<sup>29</sup>We define the three goods as in the HDI-methodology of the United Nations Development Programme (UNDP). Living standards are measured using the logarithm of GNI per capita (in 2005 US\$ PPP), health is measured by life expectancy at birth, and education by a geometric average of normalized mean years of schooling and normalized expected years of schooling. Each dimension is normalized between 0 and 1. See UNDP (2010, pp. 216-217) for details. The data have been downloaded from the UNDP website in December 2012. Note that the use of a logarithmic transformation in the definition of living standards is controversial. See Decancq, Decoster and Schokkaert (2009) and Ravallion (2012) for critical discussions.

<sup>30</sup>Noorbakhsh (2007) applies the unidimensional Gini index to the HDI values, clearly a two-stage approach. Decancq, Decoster and Schokkaert (2009) contrast the normative and two-stage approaches in the context of global well-being inequality.

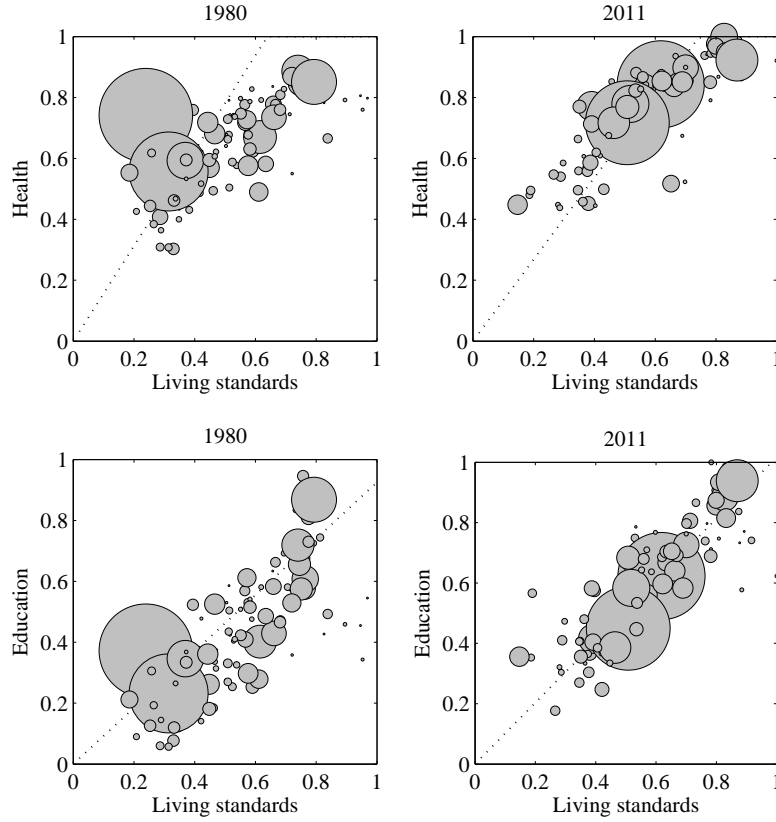


Figure 5. Living standards and health (top) and living standards and education (bottom) in 1980 (left) and 2011 (right)

points correspond to more highly populated countries. The dashed line in each panel, which connects the origin with the average bundle, has the same interpretation as the dashed line in Figure 2. Efficiency appears to have improved between 1980 and 2011, as points have moved closer to the dashed line. Equity also seems to have improved, as highly populated countries that traditionally were poorer, such as China and India, have moved upward along the dashed line.

Let us first consider the results for a specific choice of the parameters. We let  $\beta = 1$ , as in the new formulation of the HDI, and  $\alpha = 0.5$ , reflecting a moderate degree of inequity aversion. Figure 6 shows the evolution of inequality, inefficiency and inequity for these parameter values.<sup>31</sup> There is

<sup>31</sup>The decomposition presented in the figure is approximately additive. Recall that

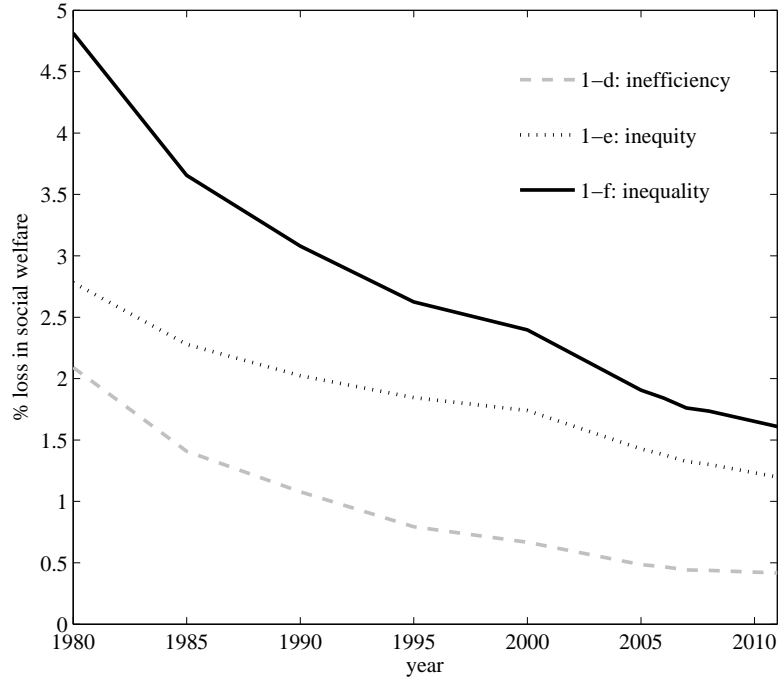


Figure 6. Inequality, inequity and inefficiency between 1980 and 2011 ( $\alpha = 0.5$ ,  $\beta = 1$ )

a clear decrease of inequality, with the loss in social welfare dropping from 4.8% in 1980 to 1.6% in 2011. Note that a value of 1.6% is only seemingly low because of the way the three goods are constructed. In absolute terms this percentage corresponds to a considerable welfare waste: for a country with the average bundle in 2011, it is equivalent to a loss of \$1,335 out of \$7,922 (keeping the health and education levels fixed). The decreasing trends for the inefficiency and inequity components are in line with the first impression from Figure 5. Note that, for the chosen values of  $\alpha$  and  $\beta$ , inefficiency falls at a considerably faster rate than inequity.

Table 1 presents results for a variety of combinations of the parameters  $\alpha$  and  $\beta$  for the years 1980 and 2011. For all considered values of the parameters, inequality, inefficiency and inequity decrease. The relative

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the decomposition is multiplicative, i.e.,  $f(X) = d(X) \times e(X)$ . Hence, the logarithm can be decomposed additively, i.e.,  $\ln f(X) = \ln d(X) + \ln e(X)$ . For values of  $t$  close to 1, we indeed have  $\ln t \approx 1 - t$ , so that  $1 - f(X) \approx 1 - d(X) + 1 - e(X)$ .

Table 1. Inequality, inefficiency and inequity in 1980 and 2011

$\alpha$	$\beta$	1980			2011		
		Inequality	Inefficiency	Inequity	Inequality	Inefficiency	Inequity
0	0	0.00	0.00	0.00	0.00	0.00	0.00
	0.5	1.03	1.03	0.00	0.21	0.21	0.00
	1	2.09	2.09	0.00	0.42	0.42	0.00
	2	4.02	4.02	0.00	0.80	0.80	0.00
	5	7.26	7.26	0.00	1.59	1.59	0.00
0.5	0	2.22	0.00	2.22	1.08	0.00	1.08
	0.5	3.50	1.03	2.49	1.34	0.21	1.14
	1	4.81	2.09	2.78	1.61	0.42	1.20
	2	7.28	4.02	3.40	2.12	0.80	1.33
	5	11.66	7.26	4.74	3.20	1.59	1.63
1	0	4.34	0.00	4.34	2.18	0.00	2.18
	0.5	5.83	1.03	4.84	2.51	0.21	2.30
	1	7.38	2.09	5.40	2.84	0.42	2.44
	2	10.35	4.02	6.60	3.50	0.80	2.71
	5	15.76	7.26	9.16	4.92	1.59	3.38
2	0	8.21	0.00	8.21	4.48	0.00	4.48
	0.5	10.05	1.03	9.11	4.96	0.21	4.76
	1	12.00	2.09	10.12	5.46	0.42	5.06
	2	15.85	4.02	12.32	6.47	0.80	5.71
	5	23.04	7.26	17.01	8.86	1.59	7.39
5	0	17.31	0.00	17.31	12.48	0.00	12.48
	0.5	19.82	1.03	18.98	13.65	0.21	13.47
	1	22.78	2.09	21.13	15.01	0.42	14.65
	2	30.10	4.02	27.18	18.24	0.80	17.58
	5	45.23	7.26	40.94	27.32	1.59	26.15

drop in efficiency is more pronounced than the relative drop in inequity across different parameter values (except of course for  $\beta = 0$ , in which case efficiency is zero). As  $\beta$  increases, the relative decrease in inefficiency is stronger. As  $\alpha$  increases, the share of inequity in inequality increases, a consequence of the fact that increasing  $\alpha$  increases inequity, while leaving inefficiency unaffected. The relative decrease in inequity falls as  $\alpha$  increases. While cases where inequality and inequity move in opposite directions do not occur in the table, such cases do occur in the data. For example, given  $\beta = 0.33$  and a high degree of inequality aversion  $\alpha = 10$ , inequality falls between 1980 and 2005, while inequity rises. This serves as an illustration that one should be cautious in identifying inequity and inequality in empirical research.

## 6 Conclusion

Normative indices of multidimensional inequality capture both the inequity and inefficiency exhibited by a distribution. Using a suggestion by Graaff (1977), we have provided a decomposition of normative inequality into its inequity and inefficiency parts for a generic class of social welfare functions.

Our analysis has revealed an intimate link between the normative approach and the two-stage approach. This is striking, as the literature regards these two approaches as inherently different. While the two-stage approach in its original conception by Maasoumi (1986) does not rest on axiomatic foundations, we have shown that it nevertheless has a solid justification within the axiomatic normative approach. A plausible view on inequality measurement may be that it should not be concerned with inefficiency at all, making the measurement of inequity the only objective. If this view is taken, then, as our analysis has shown, two-stage inequality measures are the appropriate measures to use.

The decomposition has also yielded new insights into the two main multidimensional transfer principles, viz., uniform majorization and correlation increasing majorization. These principles propagate transfers that have both equity and efficiency effects, making the overall implication of these transfers difficult to evaluate a priori. In particular, if again the view is taken that the objective is to measure inequity and not inefficiency, then uniform majorization is a clearly unappealing requirement, as it in some cases recommends transfers that unambiguously increase inequity, an intuition already apparent in Dardanoni (1995). We believe that the basic issue of how to satisfactorily generalize the simple unidimensional Pigou-Dalton principle to the multidimensional setting remains unsettled. We leave the issue to further research.

## References

- Atkinson AB, 1970. On the measurement of inequality. *Journal of Economic Theory* **2**, 244-263.
- Atkinson AB, 2003. Multidimensional deprivation: contrasting social welfare and counting approaches. *Journal of Economic Inequality* **1**, 51-65.
- Atkinson AB, Bourguignon F, 1982. The comparison of multidimensioned distributions of economic status. *Review of Economic Studies* **49**, 183-201.
- Bosmans K, Decancq K, Ooghe E, 2013. Who's afraid of aggregating money metrics? *Mimeo*.
- Bosmans K, Lauwers L, Ooghe E, 2009. A consistent multidimensional Pigou-Dalton transfer principle. *Journal of Economic Theory* **144**, 1358-1371.

- Bourguignon F, 1999. Comment on ‘Multidimensioned approaches to welfare analysis’ by E. Maasoumi. In: Silber J (ed) *Handbook of Income Inequality Measurement*. London: Kluwer Academic.
- Bourguignon F, Chakravarty SR, 2003. The measurement of multidimensional poverty *Journal of Economic Inequality* **1**, 25-49.
- Dalton H, 1920. The measurement of the inequality of incomes. *Economic Journal* **30**, 348-361.
- Dardanoni V, 1995. On multidimensional inequality measurement. In: Dagum C, Lemmi A (eds) *Income Distribution, Social Welfare, Inequality and Poverty, Research on Economic Inequality*, vol. 6, Stamford, CT: JAI Press.
- Debreu G, 1951. The coefficient of resource utilization. *Econometrica* **19**, 273-292.
- Debreu G, 1976. Least concave utility functions. *Journal of Mathematical Economics* **3**, 121-129.
- Decancq K, Decoster A, Schokkaert E, 2009. The evolution of world inequality in well-being. *World Development* **37**, 11-25.
- Decancq K, Fleurbaey M, Maniquet F, 2012. Multidimensional poverty measurement: Shouldn’t we take preferences into account? *Mimeo*.
- Decancq K, Ooghe E, 2010. Has the world moved forward? A robust multidimensional evaluation. *Economics Letters* **107**, 266-269.
- Duclos J-Y, Sahn DE, Younger SD, 2011. Partial multidimensional inequality orderings. *Journal of Public Economics* **95**, 225-238.
- Fleurbaey M, 2006. Social welfare, priority to the worst-off and the dimensions of individual well-being. In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.
- Gajdos T, Weymark JA, 2005. Multidimensional generalized Gini indices. *Economic Theory* **26**, 471-496.
- Graaff J de V, 1977. Equity and efficiency as components of the general welfare. *South African Journal of Economics* **45**, 362-375.
- Kannai Y, 1980. The ALEP definition of complementarity and least concave utility functions. *Journal of Economic Theory* **22**, 115-117.
- Kihlstrom RE, Mirman LJ, 1974. Risk aversion with many commodities. *Journal of Economic Theory* **8**, 361-368.
- Kihlstrom RE, Mirman LJ, 1981. Constant, increasing and decreasing risk aversion with many commodities *Review of Economic Studies* **48**, 271-280.
- Kolm S-Ch, 1969. The optimal production of social justice. In: Margolis J, Guitton H (eds) *Public Economics*. London: Macmillan.
- Kolm S-Ch, 1977. Multidimensional egalitarianisms. *Quarterly Journal of Economics* **91**, 1-13.
- Lasso de la Vega MC, Urrutia A, 2011. Characterizing how to aggregate the individuals deprivations in a multidimensional framework. *Journal of Economic Inequality* **9**, 183-194.
- Maasoumi E, 1986. The measurement and decomposition of multi-dimensional inequality. *Econometrica* **54**, 991-998.



- Noorbakhsh F, 2007. International convergence or higher inequality in human development? Evidence for 1975-2002. In: Mavrotas G, Shorrocks T (eds) *Advancing Development: Core Themes in Global Economics*. Basingstoke, New York: Palgrave Macmillan.
- Ravallion M, 2012. Troubling tradeoffs in the Human Development Index. *Journal of Development Economics* **99**, 201-209.
- Sen AK, 1973. *On Economic Inequality*. Oxford: Clarendon Press.
- Seth S, 2011. A class of association sensitive multidimensional welfare indices. Forthcoming in *Journal of Economic Inequality*.
- Trannoy A, 2006. Multidimensional egalitarianism and the dominance approach: a lost paradise? In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.
- Tsui K-Y, 1995. Multidimensional generalizations of the relative and absolute inequality indices: the Atkinson-Kolm-Sen approach. *Journal of Economic Theory* **67**, 251-265.
- Tsui K-Y, 1999. Multidimensional inequality and multidimensional generalized entropy measures: an axiomatic derivation. *Social Choice and Welfare* **16**, 145-157.
- United Nations Development Programme, 2010. *Human Development Report 2010*. New York: Oxford University Press.
- Weymark JA, 2006. The normative approach to the measurement of multidimensional inequality. In: Farina F, Savaglio E (eds) *Inequality and Economic Integration*. London: Routledge.

## Appendix A

We provide a characterization of the class of social welfare functions  $\mathcal{W}$ . First, we formally define the seven properties presented in Section 2. We write  $X > Y$  if  $x_{ij} \geq y_{ij}$  for all individuals  $i$  and all goods  $k$  with at least one inequality holding strictly. We use  $1_{n \times m}$  to denote the  $n \times m$  matrix with a one at each entry.

- (i) *Anonymity*. For all distributions  $X$  and  $Y$  in  $D$ , if  $X$  and  $Y$  are equal up to a rearrangement of rows, then  $W(X) = W(Y)$ .
- (ii) *Monotonicity*. For all distributions  $X$  and  $Y$  in  $D$ , if  $X > Y$ , then  $W(X) > W(Y)$ .
- (iii) *Continuity*. The function  $W$  is continuous.
- (iv) *Weak uniform majorization*. For all distributions  $X$  and  $Y$  in  $D$ , if  $X = QY$  with  $Q$  a bistochastic matrix that is not a permutation matrix, then  $W(X) \geq W(Y)$ .
- (v) *Normalization*. For each real number  $\lambda$ , we have  $W(\lambda 1_{n \times m}) = \lambda$ .
- (vi) *Individualism*. There exist a function  $U_i : B \rightarrow \mathbb{R}$  for each individual  $i$  and a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for each distribution  $X$  in  $D$ , we have  $W(X) = V(U_1(x_1), U_2(x_2), \dots, U_n(x_n))$ .
- (vii) *Homotheticity*. For all distributions  $X$  and  $Y$  in  $D$  and each positive real number  $\lambda$ , we have  $W(X) \geq W(Y)$  if and only if  $W(\lambda X) \geq W(\lambda Y)$ .

We now prove the following result.

*A social welfare function  $W$  satisfies properties (i) to (vii) if and only if there exists a continuous, concave, linearly homogenous and strictly increasing function  $u : B \rightarrow \mathbb{R}$  with  $u(1_m) = 1$  and a continuous, Schur-concave, linearly homogenous and strictly increasing function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $v(1_n) = 1$  such that, for each distribution  $X$  in  $D$ , we have  $W(X) = v(u(x_1), u(x_2), \dots, u(x_n))$ .*

*Proof.* Let  $W$  be a social welfare function that satisfies properties (i) to (vii). By anonymity, monotonicity, continuity and individualism, there exist a continuous and strictly increasing function  $\hat{u} : B \rightarrow \mathbb{R}$  and a continuous and strictly increasing function  $\hat{v} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for each distribution  $X$  in  $D$ , we have  $W(X) = \hat{v}(\hat{u}(x_1), \hat{u}(x_2), \dots, \hat{u}(x_n))$ .

Homotheticity implies that, for all bundles  $x$  and  $y$  in  $B$  and each positive real number  $\lambda$ , we have  $W((x, x, \dots, x)') \geq W((y, y, \dots, y)')$  if and only if  $W((\lambda x, \lambda x, \dots, \lambda x)') \geq W((\lambda y, \lambda y, \dots, \lambda y)')$ . It follows that, for all bundles  $x$  and  $y$  in  $B$  and each positive real number  $\lambda$ , we have  $\hat{u}(x) \geq \hat{u}(y)$

if and only if  $\hat{u}(\lambda x) \geq \hat{u}(\lambda y)$ , i.e.,  $\hat{u}$  is a homothetic function. Hence, there exists a linearly homogenous function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $u(1, 1, \dots, 1) = 1$  and a strictly increasing function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\hat{u} = \psi \circ u$ . Let the function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  be such that, for each  $(s_1, s_2, \dots, s_n)$  in  $\mathbb{R}^n$ , we have  $\hat{v}(s_1, s_2, \dots, s_n) = v(\psi^{-1}(s_1), \psi^{-1}(s_2), \dots, \psi^{-1}(s_n))$ . It follows that, for each distribution  $X$  in  $D$ , we have  $W(X) = v(u(x_1), u(x_2), \dots, u(x_n))$ . The functions  $u$  and  $v$  inherit continuity and strict increasingness from the functions  $\hat{u}$  and  $\hat{v}$ . Furthermore, by normalization,  $v$  is linearly homogenous and  $v(1, 1, \dots, 1) = 1$ . What remains to be shown is that  $u$  is concave and that  $v$  is Schur-concave.

We first show that  $u$  is quasi-concave. Seeking a contradiction, suppose that  $u$  is not quasi-concave. Then there exists a bundle  $x$  in  $B$  such that the upper contour set  $\{y \in B | u(y) \geq u(x)\}$  is not convex. This implies that there exist two bundles  $a$  and  $b$  in  $B$  such that  $u(0.5a + 0.5b) < u(a) = u(b)$ . Let  $T$  be an  $n \times n$  matrix, where each row  $k \neq i, j$  has 1 at the  $k$ th entry and 0 at each other entry, and where rows  $i$  and  $j$  have 0.5 at the  $i$ th entry, 0.5 at the  $j$ th entry and 0 at each other entry. Let  $X$  be a distribution in  $D$  with bundle  $a$  at the  $i$ th row and bundle  $b$  at the  $j$ th row. Distribution  $TX$  has bundle  $0.5a + 0.5b$  at both the  $i$ th and the  $j$ th rows and coincides with  $X$  on all other rows. Since  $u(0.5a + 0.5b) < u(a) = u(b)$ , we have  $W(X) > W(TX)$ . But weak uniform majorization implies  $W(X) \geq W(TX)$  because  $T$  is a bistochastic matrix that is not a permutation matrix. We have a contradiction and hence  $u$  is quasi-concave. We know from Kihlstrom and Mirman (1981) that  $u$  is a least concave representation since it is linearly homogenous in addition to being continuous, strictly increasing and quasi-concave. Hence,  $u$  is concave.

Next we show that  $v$  is Schur-concave. Let  $x(t)$  denote the bundle  $(t, t, \dots, t)$  in  $B$ . Jointly, anonymity and weak uniform majorization imply that, for each  $(t_1, t_2, \dots, t_n)$  in  $\mathbb{R}^n$  and each bistochastic matrix  $Q$ , we have  $W((x(t_1), x(t_2), \dots, x(t_n))') \leq W(Q(x(t_1), x(t_2), \dots, x(t_n))')$ . Using that  $u$  is linearly homogenous function and  $u(1, 1, \dots, 1) = 1$ , it follows that, for each  $(t_1, t_2, \dots, t_n)$  in  $\mathbb{R}^n$  and each bistochastic matrix  $Q$ , we have  $v((t_1, t_2, \dots, t_n)') \leq v(Q(t_1, t_2, \dots, t_n)')$ . Hence,  $v$  is Schur-concave.  $\square$

## Appendix B

We provide a concise treatment of the analysis of Section 3 for the absolute case. We omit all explanations that are similar to those in Section 3. The only change we make to the basic assumptions in Section 2 is that good amounts can be zero or negative in addition to positive.

We focus on the class of social welfare functions  $\mathcal{W}^*$ . A social welfare function  $W$  is in  $\mathcal{W}^*$  if and only if there exist a continuous, concave, unit translatable<sup>32</sup> and strictly increasing function  $u : B \rightarrow \mathbb{R}$  with  $u(1_m) = 1$  and a continuous, Schur-concave, unit translatable and strictly increasing function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $v(1_n) = 1$  such that, for each distribution  $X$  in  $D$ , we have  $W(X) = v(u(x_1), u(x_2), \dots, u(x_n))$ .

The class  $\mathcal{W}^*$  is characterized by the combination of properties (i) to (vi) in Appendix A and the following property of translatability (we omit the proof).

*Translatability.* For all distributions  $X$  and  $Y$  in  $D$  and each positive real number  $\lambda$ , we have  $W(X) \geq W(Y)$  if and only if  $W(X + \lambda 1_{n \times m}) \geq W(Y + \lambda 1_{n \times m})$ .

The definitions of inequality and the inefficiency and inequity components for the absolute case are as follows.

*Definition 1\*.* Let  $X$  be a distribution in  $D$ . Let  $f^*(X)$  be the largest real number for which there exists a distribution  $Y$  in  $D$  such that  $W(Y) = W(X)$  and  $\mu_Y = \mu_X - f^*(X)1_n$ . Then,  $f^*(X)$  is the inequality level of distribution  $X$ .

*Definition 2\*.* Let  $X$  be a distribution in  $D$ . Let  $d^*(X)$  be the largest real number for which there exists a distribution  $Z$  in  $D$  such that  $u(z_i) = u(x_i)$  for each individual  $i$  and  $\mu_Z = \mu_X - d^*(X)1_n$ . Then,  $d^*(X)$  is the inefficiency level of distribution  $X$ .

*Definition 3\*.* Let  $X$  be a distribution in  $D$ . Let  $Z$  be a distribution in  $D$  obtained from distribution  $X$  as in Definition 2\*. Let  $e(X)$  be the largest real number for which there exists a distribution  $T$  such that  $W(T) = W(Z)$  and  $\mu_T = \mu_Z - e^*(X)1_n$ . Then,  $e^*(X)$  is the inequity level of distribution  $X$ .

We obtain an additive decomposition of inequality into inefficiency and inequity parts. That is, for each distribution  $X$  in  $D$ , we have

$$f^*(X) = d^*(X) + e^*(X).$$

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<sup>32</sup>A function  $\psi : \mathbb{R}^\ell \rightarrow \mathbb{R}$  is unit translatable if  $\psi(s + \lambda 1_\ell) = \lambda + \psi(s)$  for each  $s$  in  $\mathbb{R}^\ell$  and each real number  $\lambda$ .

For a distribution  $X$  in  $D$ , inequality  $f^*(X)$  is defined by  $W(X) = W(X_\mu - f^*(X)1_{n \times m})$ . Using unit translatability of  $W$ , we obtain

$$f^*(X) = W(X_\mu) - W(X).$$

For each individual  $i$ , define  $t_i$  by  $u(x_i) = u(\mu_X - t_i)$ . The efficient distribution  $Z$  equals  $(\mu_X - t_1, \mu_X - t_2, \dots, \mu_X - t_n)'$ . Since  $\sum_{i=1}^n \mu_X - t_i = n\mu_Z$ , we have  $d^*(X) = \sum_{i=1}^n t_i/n$ . Moreover,  $t_i = u(\mu_X) - u(x_i)$  because  $u$  is unit translatable. We obtain

$$d^*(X) = u(\mu_X) - \frac{1}{n} \sum_{i=1}^n u(x_i).$$

Finally, using  $e^* = f^* - d^*$  and  $u(\mu_X) = W(X_\mu)$ , we obtain

$$e^*(X) = \frac{1}{n} \sum_{i=1}^n u(x_i) - w(X).$$

## Appendix C

*Table 2.* Countries in the data set

Afghanistan	Germany	Nicaragua
Algeria	Ghana	Niger
Argentina	Greece	Norway
Australia	Guatemala	Pakistan
Austria	Guyana	Panamá
Bahrain	Haiti	Papua New Guinea
Bangladesh	Honduras	Paraguay
Belgium	Hong Kong	Peru
Belize	Hungary	Philippines
Benin	Iceland	Portugal
Bolivia	India	Qatar
Botswana	Indonesia	Republic of Korea
Brazil	Iran	Rwanda
Brunei Darussalam	Ireland	Saudi Arabia
Burundi	Israel	Senegal
Cameroon	Italy	Sierra Leone
Canada	Jamaica	South Africa
Central African Republic	Japan	Spain
Chile	Jordan	Sri Lanka
China	Kenya	Sudan
Colombia	Kuwait	Sweden
Congo	Lesotho	Switzerland
Costa Rica	Luxembourg	Syria
Côte d'Ivoire	Malawi	Thailand
Democratic Republic of the Congo	Malaysia	Togo
Denmark	Mali	Trinidad and Tobago
Dominican Republic	Malta	Tunisia
Ecuador	Mauritania	Turkey
Egypt	Mauritius	United Arab Emirates
El Salvador	Mexico	United Kingdom
Fiji	Morocco	United States
Finland	Myanmar	Uruguay
France	Nepal	Venezuela
Gabon	Netherlands	Zambia
Gambia	New Zealand	Zimbabwe

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